Chapter 4

An Introduction to

Numerical Methods for Parabolic Equations

First Session Contents:

1) Introduction
2) Finite Difference Methods
3) Explicit methods for Parabolic Equations
4) Truncation Error (T.E.)
5) Consistency
6) Stability

Parabolic PDE

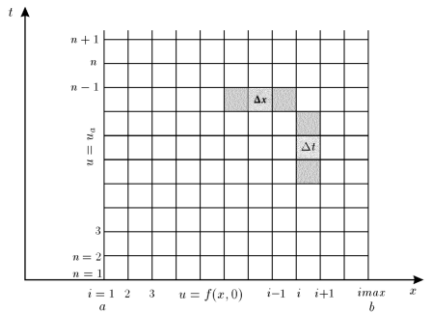
In this chapter, we will discuss about the numerical solution of one-dimensional parabolic PDE as given below:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad a < x < b$$

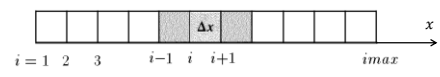
B.C.'s $\begin{cases} u(a,t) = u_a(t) \\ u(b,t) = u_b(t) \end{cases}$

I.C. $u(x,0) = f(x)$

Computational Domain for Parabolic PDEs



Computational Domain for Parabolic PDEs

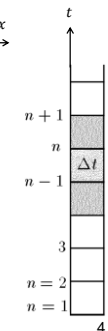


Grid Points or Mesh Points

$$x_i = (i-1)\Delta x = (i-1)h \quad i = 1, 2, 3, \ldots imax$$
$$t_n = (n-1)\Delta t = (n-1)k \quad n = 1, 2, 3, \ldots$$

$k = \Delta t$ **Time Step**

$h = \Delta x$ **Grid Spacing**



Discretization

General PDE $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

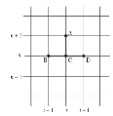
Transient Term $\left.\dfrac{\partial u}{\partial t}\right|_i^n = \dfrac{u_i^{n+1} - u_i^n}{k} + O(k)$ (Forward Difference)

Diffusion Term $\left.\dfrac{\partial^2 u}{\partial x^2}\right|_i^n = \dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + O(h^2)$ (Central Difference)

Substituting transient and diffusion terms in the PDE, we have

$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + O(k, h^2)$$

$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad \text{where} \quad r = \frac{k}{h^2}$$



$$u_i^{n+1} = u_i^n + r\delta_x^2 u_i^n = (1 + r\delta_x^2)u_i^n \quad \text{where} \quad \delta_x \text{ is the central difference operator}$$

Explicit Methods for Parabolic PDEs

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$ 1-D transient Heat Conduction Problem

Using the following Eq., the temperature at time step n+1 can be obtained from the temperature at time step n.

$$u_i^{n+1} = ru_{i+1}^n + (1 - 2r)u_i^n + ru_{i-1}^n \quad , \quad r = \frac{k}{h^2} \quad , \quad \text{T.E.} = O(k, h^2)$$

At time step n=1, temperature is known as initial condition.
Therefore,
Temperature at n+1, n+2, … can be obtained.
$[u^{n+1}]=[A][u^n]$

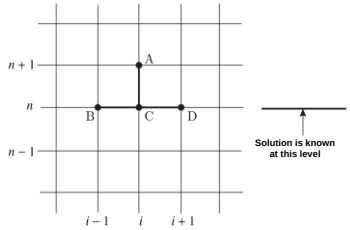
A method which calculates the state of a system at a later time from the state of the system at the current time is called
Explicit Method

Explicit Method (FTCS)

Transient Term (Forward Difference)
Diffusion Term (Central Difference)

$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

FTCS
Forward **T**ime **C**entral **S**pace

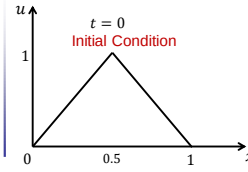


Example (FTCS)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1$$

Initial Condition $u(x,0) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$

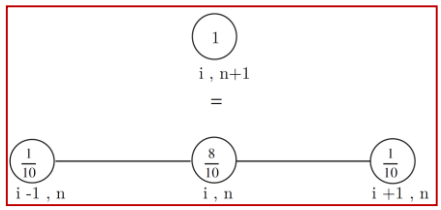
Boundary Conditions $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$



$$h = \frac{1}{10} \qquad k = \frac{1}{1000} \qquad r = \frac{k}{h^2} = \frac{1}{10}$$

$$u_i^{n+1} = \frac{1}{10}(u_{i-1}^n + 8u_i^n + u_{i+1}^n)$$

Example (FTCS)



$$u_i^{n+1} = \frac{1}{10}(u_{i-1}^n + 8u_i^n + u_{i+1}^n)$$

Example (FTCS)

Numerical Solution

$$u_i^{n+1} = \frac{1}{10}(u_{i-1}^n + 8u_i^n + u_{i+1}^n)$$

$$u_6^2 = \frac{1}{10}(u_5^1 + 8 \times u_6^1 + u_7^1) = \frac{1}{10}[0.8 + (8 \times 1) + 0.8] = 0.9600$$

$$u_5^3 = \frac{1}{10}(u_4^2 + 8 \times u_5^2 + u_6^2) = \frac{1}{10}[0.6 + (8 \times 0.8) + 0.96] = 0.7960$$

Exact Solution

$$u = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(\sin\frac{n\pi}{2})(\sin n\pi x)\exp(-n^2\pi^2 t)$$

Example (FTCS)

$$r = \frac{k}{h^2} = 0.10$$

Time	$i=1$ $x=0$	$i=2$ $x=0.1$	$i=3$ $x=0.2$	$i=4$ $x=0.3$	$i=5$ $x=0.4$	$i=6$ $x=0.5$	$i=7$ $x=0.6$
0.000	0	0.2000	0.4000	0.6000	0.8000	1.0000	0.8000
0.001	0	0.2000	0.4000	0.6000	0.8000	0.9600	0.8000
0.002	0	0.2000	0.4000	0.6000	0.7960	0.9280	0.7960
0.003	0	0.2000	0.4000	0.5996	0.7896	0.9016	0.7896
0.004	0	0.2000	0.4000	0.5986	0.7818	0.8792	0.7818
0.005	0	0.2000	0.3999	0.5971	0.7732	0.8597	0.7732
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.01	0	0.1996	0.3968	0.5822	0.7281	0.7867	0.7281
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.02	0	0.1938	0.3781	0.5373	0.6486	0.6891	0.6486

Example (FTCS)

$$r = \frac{k}{h^2} = 0.10$$

Time	$i=1$ $x=0$	$i=2$ $x=0.1$	$i=3$ $x=0.2$	$i=4$ $x=0.3$	$i=5$ $x=0.4$	$i=6$ $x=0.5$	$i=7$ $x=0.6$
0.000	0	0.2000	0.4000	0.6000	0.8000	1.0000	0.8000
0.001	0	0.2000	0.4000	0.6000	0.8000	0.9600	0.8000
0.002	0	0.2000	0.4000	0.6000	0.7960	0.9280	0.7960
0.003	0	0.2000	0.4000	0.5996	0.7896	0.9016	0.7896
0.004	0	0.2000	0.4000	0.5986	0.7818	0.8792	0.7818
0.005	0	0.2000	0.3999	0.5971	0.7732	0.8597	0.7732
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.01	0	0.1996	0.3968	0.5822	0.7281	0.7867	0.7281
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.02	0	0.1938	0.3781	0.5373	0.6486	0.6891	0.6486

Symmetry

Example (FTCS)

	Time	Numerical Solution ($x = 0.3$)	Exact Solution ($x = 0.3$)	Absolute Error	Relative Error %
$x = 0.30$	$t = 0.005$	0.5971	0.5966	0.0005	0.08
	$t = 0.01$	0.5822	0.5799	0.0023	0.4
	$t = 0.02$	0.5373	0.5334	0.0039	0.7
	$t = 0.1$	0.2472	0.2444	0.0028	1.1

	Time	Numerical Solution ($x = 0.5$)	Exact Solution ($x = 0.5$)	Absolute Error	Relative Error %
$x = 0.50$	$t = 0.005$	0.8597	0.8404	0.0193	2.3
	$t = 0.01$	0.7867	0.7743	0.0124	1.6
	$t = 0.02$	0.6891	0.6809	0.0082	1.2
	$t = 0.1$	0.3056	0.3021	0.0035	1.2

13

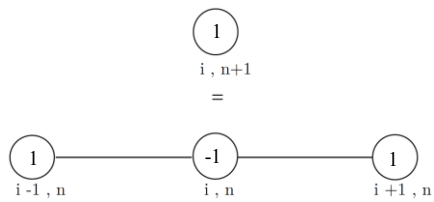
Example (FTCS)

	Time	Numerical Solution ($x = 0.3$)	Exact Solution ($x = 0.3$)	Absolute Error	Relative Error %
$x = 0.30$	$t = 0.005$	0.5971	0.5966	0.0005	0.08
	$t = 0.01$	0.5822	0.5799	0.0023	0.4
	$t = 0.02$	0.5373	0.5334	0.0039	0.7
	$t = 0.1$	0.2472	0.2444	0.0028	1.1

	Time	Numerical Solution ($x = 0.5$)	Exact Solution ($x = 0.5$)	Absolute Error	Relative Error %
$x = 0.50$	$t = 0.005$	0.8597	0.8404	0.0193	2.3
	$t = 0.01$	0.7867	0.7743	0.0124	1.6
	$t = 0.02$	0.6891	0.6809	0.0082	1.2
	$t = 0.1$	0.3056	0.3021	0.0035	1.2

14

Example (FTCS)



$$h = \frac{1}{10} \qquad k = \frac{1}{100} \qquad r = \frac{k}{h^2} = 1$$

$$u_i^{n+1} = u_{i-1}^n - u_i^n + u_{i+1}^n$$

15

Example (FTCS)

$r = 0.10$

t	$i = 1$ $x = 0$	$i = 2$ $x = 0.1$	$i = 3$ $x = 0.2$	$i = 4$ $x = 0.3$	$i = 5$ $x = 0.4$	$i = 6$ $x = 0.5$	$i = 7$ $x = 0.6$
0.000	0	0.2000	0.4000	0.6000	0.8000	1.0000	0.8000
0.001	0	0.2000	0.4000	0.6000	0.8000	0.9600	0.8000
0.002	0	0.2000	0.4000	0.6000	0.7960	0.9280	0.7960
0.003	0	0.2000	0.4000	0.5996	0.7896	0.9016	0.7896
0.004	0	0.2000	0.4000	0.5986	0.7818	0.8792	0.7818
0.005	0	0.2000	0.3999	0.5971	0.7732	0.8597	0.7732
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.01	0	0.1996	0.3968	0.5822	0.7281	0.7867	0.7281
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.02	0	0.1938	0.3781	0.5373	0.6486	0.6891	0.6486

$r = 1$

t	$i = 1$ $x = 0$	$i = 2$ $x = 0.1$	$i = 3$ $x = 0.2$	$i = 4$ $x = 0.3$	$i = 5$ $x = 0.4$	$i = 6$ $x = 0.5$	$i = 7$ $x = 0.6$
0.00	0	0.2	0.4	0.6	0.8	1.0	0.8
0.01	0	0.2	0.4	0.6	0.8	0.6	0.8
0.02	0	0.2	0.4	0.6	0.4	1.0	0.4
0.03	0	0.2	0.4	0.2	1.2	−0.2	1.2
0.04	0	0.2	0.0	1.4	−1.2	2.6	−1.2

16

Example (FTCS)

$r = 0.10$

t	$i=1$ $x=0$	$i=2$ $x=0.1$	$i=3$ $x=0.2$	$i=4$ $x=0.3$	$i=5$ $x=0.4$	$i=6$ $x=0.5$	$i=7$ $x=0.6$
0.000	0	0.2000	0.4000	0.6000	0.8000	1.0000	0.8000
0.001	0	0.2000	0.4000	0.6000	0.8000	0.9600	0.8000
0.002	0	0.2000	0.4000	0.6000	0.7960	0.9280	0.7960
0.003	0	0.2000	0.4000	0.5996	0.7896	0.9016	0.7896
0.004	0	0.2000	0.4000	0.5986	0.7818	0.8792	0.7818
0.005	0	0.2000	0.3999	0.5971	0.7732	0.8597	0.7732
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.01	0	0.1996	0.3968	0.5822	0.7281	0.7867	0.7281
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.02	0	0.1938	0.3781	0.5373	0.6486	0.6891	0.6486

$r = 1$

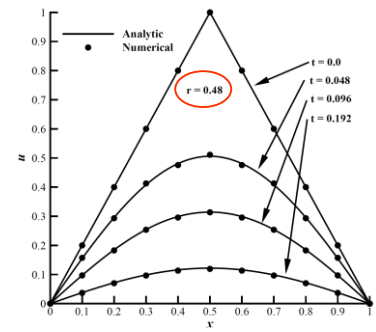
t	$i=1$ $x=0$	$i=2$ $x=0.1$	$i=3$ $x=0.2$	$i=4$ $x=0.3$	$i=5$ $x=0.4$	$i=6$ $x=0.5$	$i=7$ $x=0.6$
0.00	0	0.2	0.4	0.6	0.8	1.0	0.8
0.01	0	0.2	0.4	0.6	0.8	0.6	0.8
0.02	0	0.2	0.4	0.6	0.4	1.0	0.4
0.03	0	0.2	0.4	0.2	1.2	−0.2	1.2
0.04	0	0.2	0.0	1.4	−1.2	2.6	−1.2

The value of r plays an important role in explicit methods

$$0 < r \leq \frac{1}{2}$$

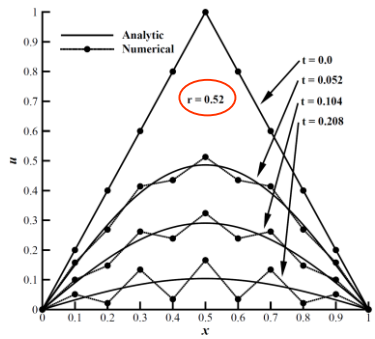
17

Example (FTCS)



18

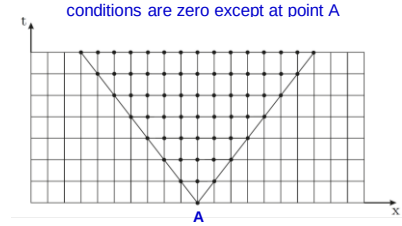
Example (FTCS)



19

Propagation Speed of a Disturbance

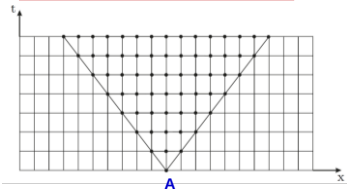
Consider a 1-D parabolic PDE with all initial conditions are zero except at point A



$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

20

Propagation Speed of a Disturbance



Slope of Characteristics lines:

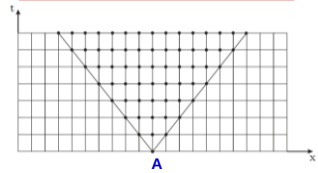
$$\frac{dt}{dx} = \frac{\Delta t}{\Delta x} \neq 0$$

The propagation speed of a disturbance in parabolic PDEs for given time is **infinity**

?

The propagation speed of a disturbance in finite-difference form of the Parabolic PDE for given time is $\Delta x / \Delta t$

Propagation Speed of a Disturbance



Slope of Characteristics lines:

$$\frac{dt}{dx} = \frac{\Delta t}{\Delta x} \neq 0$$

The propagation speed of a disturbance in parabolic PDEs for given time is **infinity**

?

The propagation speed of a disturbance in finite-difference form of the Parabolic PDE for given time is $\Delta x / \Delta t$

Solution: $\frac{\Delta x}{\Delta t} = \frac{\Delta x}{r(\Delta x)^2} = \frac{1}{r\Delta x}$ $\left\{ \begin{matrix} \Delta x \to 0 \\ r = \text{Const.} \end{matrix} \right\} \Rightarrow \frac{dt}{dx} \to 0$

Truncation Error (T.E.)

$L[u] = f \qquad (x,t) \in \Omega$ General PDE

$D[U_i^n] = f_i^n \qquad (x,t) \in \Omega$ Discretized PDE

$U_i^n \approx u_i^n = u(x_i, t_n)$ If h and k are small enough then T.E. approaches to zero and the discretized form is a good approximation of general PDE

Truncation Error is Defined as $T_i^n = D[u_i^n] - f_i^n$ as $h, k \to 0$ independently

Truncation Error (T.E.)

$$\underbrace{\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}}_{\text{PDE}} = \underbrace{\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{1}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)}_{\text{FDE}} + \underbrace{\left[-\frac{\partial^2 u}{\partial t^2}\bigg|_i^n \frac{\Delta t}{2} + \frac{\partial^4 u}{\partial x^4}\bigg|_i^n \frac{(\Delta x)^2}{12} + \dots \right]}_{\text{T.E.}}$$

T.E. is obtained by writing the Taylor series of each term around point (x_i, t_n)

Truncation Error (T.E.)

$$\underbrace{\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}}_{\text{PDE}} = \underbrace{\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{1}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)}_{\text{FDE}} + \underbrace{\left[-\frac{\partial^2 u}{\partial t^2}\bigg|_i^n \frac{\Delta t}{2} + \frac{\partial^4 u}{\partial x^4}\bigg|_i^n \frac{(\Delta x)^2}{12} + \dots \right]}_{\text{T.E.}}$$

T.E. is obtained by writing the Taylor series of each term around point (x_i, t_n)

- Is the finite difference form of PDE acceptable?
- Does the marching method give a good approximation of PDE?

The finite-difference form of PDE should satisfy both **Stability** and **Consistency** conditions

Consistency

Finite-difference form of a PDE is consistent if:

$$\lim_{h,k\to 0} \text{T.E.}_i^n = 0$$

In some finite-difference methods the T.E. is $\frac{O(\Delta x)}{O(\Delta t)}$

These methods are consistent if $\frac{\Delta x}{\Delta t} \to 0$

DuFort-Frankel method $\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{1}{(\Delta x)^2}[u_{i+1}^n - \underbrace{(u_i^{n+1} + u_i^{n-1})}_{2u_i^n} + u_{i-1}^n]$

T.E. $\leftarrow$ $+\frac{1}{12}\frac{\partial^4 u}{\partial x^4}\bigg|_i^n (\Delta x)^2 - \frac{\partial^2 u}{\partial t^2}\bigg|_i^n \left(\frac{\Delta t}{\Delta x}\right)^2 - \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\bigg|_i^n (\Delta t)^2$

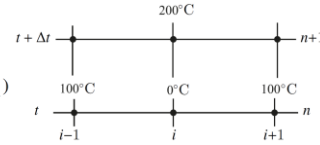
The DuFort-Frankel method is consistent if $\lim_{\Delta x, \Delta t \to 0} \left(\frac{\Delta x}{\Delta t}\right) = 0$

Stability

- Concept of stability occurs in marching problems
- A numerical method is **stable**
  If the errors of any kind (Round of error and Truncation error) will not grow (increasing unconditionally) during time marching.
- Generally, analysis of Consistency in a numerical method is more easier than analysis of Stability
- An explicit method is stable if $r = [\frac{\Delta t}{(\Delta x)^2}] \le \frac{1}{2}$

Example

Forward Time Central Space



$$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$r = 1$$

$$u_i^{n+1} = u_{i-1}^n - u_i^n + u_{i+1}^n$$

u^{n+1}=100-0+100=200

Convergence for Marching Problems

A numerical method converges if its global discretization error approaches zero as the mesh is refined

$$\lim_{h,k\to 0} | U_i^n - u_i^n | = 0 \qquad (x_i, t_n) \in \Omega$$

Generally, a numerical method **converges** if it is both **stable** and **consistence**

Lax's Equivalence Theorem

Given a well-posed initial value problem and a finite-difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence

29